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Phase spaces for quantum elementary systems in anti-de Sitter and Minkowski spacetimes

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Abstract. In this paper we give a phase space description of a massive, spin s, quantum elementary system on the anti-de Sitter spacetime. The latter is associated with a discrete series representation of the kinematical group $\mathrm{SO}_0(3,2) \simeq \mathrm{Sp}(4,\mathbf{R})/Z_2$, taken in its Fock–Bargmann realization. When the zero curvature limit (contraction) is carried out, we obtain a Poincaré quantum elementary system in its momentum representation (i.e. the usual Wigner representation), at the expense of imposing a polarization condition. This polarization appears as a consequence of the contraction procedure, and it is imposed in order to avoid the appearance of singular terms in the contracted generators.

1. Introduction

Very few fundamental physical constants are ultimately necessary to deal with a massive quantum elementary system at the kinematical level: a mass scale, say m, the fundamental speed c, the elementary action $\hbar$, and finally a length scale, that will be denoted κ^{-1} consistently with the spirit of the present paper. These are necessary and sufficient to build up the dimensionless quantity,

$$\xi = \frac{\hbar\kappa}{mc} \tag{1.1}$$

proper to the system. They also enable us to travel from one *physics* to another one through contraction/deformation procedures. Namely [1], c for connecting Einsteinian and Galilean physics, m for connecting mobile and static physics, κ for connecting (anti-) de Sitterian and Einsteinian physics, and $\hbar$ for connecting quantum and classical physics. The three first ones are contraction/deformation parameters in a group theoretical context [2, 3], whereas the last one is a deformation parameter starting from the symplectic structure of classical mechanics [4].

Figure 1 is extracted from the Bacry and Lévy-Leblond paper [1] where eleven possible kinematics were originally listed. We have selected the seven kinematics that appear to have a reasonable physical content. Two of them are of *maximal*

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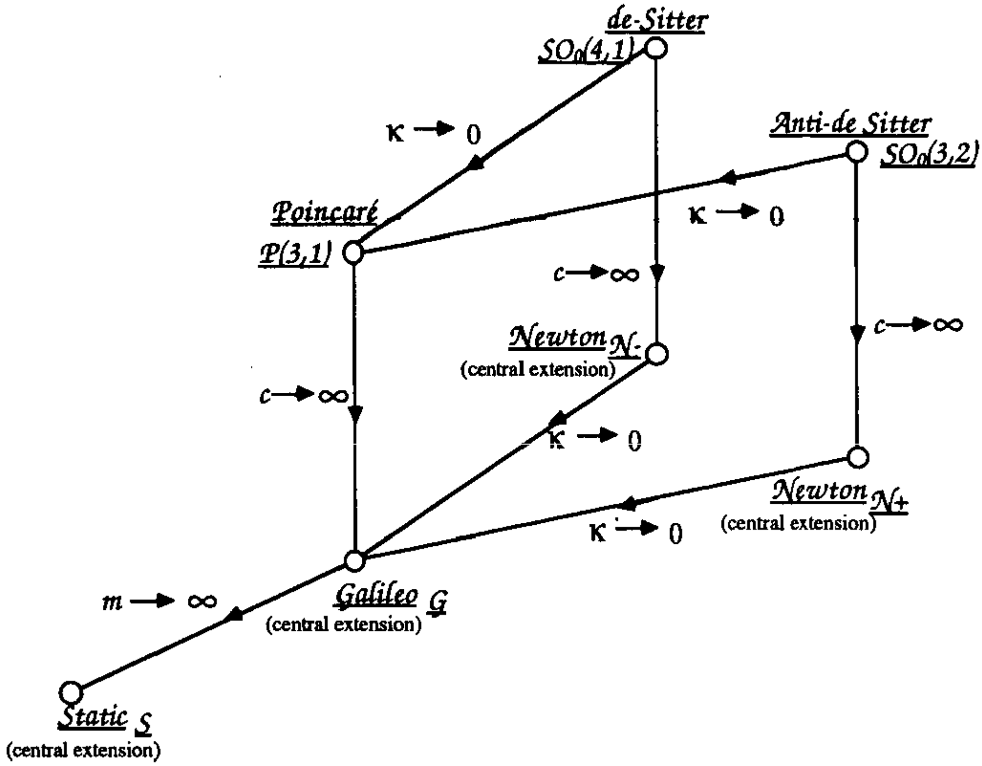

Figure 1. Contraction–deformation relationships between four-dimensional spacetime relativities. The related kinematical groups are respectively the two de Sitterian groups $SO_0(4,1)$ and $SO_0(3,2)$, the (proper orthochronous) Poincaré group $\mathcal{P}_+^\uparrow(3,1)$, the two Newton groups $\mathcal{N}_\pm$, the Galileo group $\mathcal{G}$ and finally the static group $\mathcal{S}$. At each step of a contraction (de Sitter $\longmapsto$ static) some of the original ten dimensionless group parameters aquire a physical dimension, e.g. become *length-like*, *time-like* or *momentum-like*. Correspondingly a part of the simple structure of the original group breaks down into a semi-direct product structure.

symmetry, i.e. their kinematical groups are the anti-de Sitterian pseudo-orthogonal groups $SO_0(4,1)$ and $SO_0(3,2)$ and no physical unit is necessary to standardize their ten (pseudo-) angular parameters. They are departures for successive contractions until reaching the ultimate *kinematics* where nothing moves. At each step, some of the parameters acquire a physical dimension. They may become *length-like*, *time-like* or *momentum-like*. Correspondingly the simple-group structure breaks down into a semi-direct one.

Quantum elementary systems are associated with (projective) unitary, irreducible representations (UIR) of the (possibly extended) kinematic group (or one of its covering). Wigner originated this point of view in his famous 1939 paper [5] where an (Einsteinian) elementary system of mass m and spin s is shown to be identified with the representation $\mathcal{P}(m,s)$ of the Poincaré group $\mathcal{P}_+^\uparrow(3,1)$. He was followed by Inönü [6], Lévy-Leblond [7] and Voisin [8] who applied the Wigner ideas to Galilean systems, and by Gürsey [9] and Fronsdal [10] who extended them to de Sitterian and anti-de Sitterian systems respectively.

The contraction procedure depends on the physical surroundings. Separate physical quantities may become singular while some of their combinations acquire a definite physical meaning in the new paradigm. The way we calculate the limit is also strongly dependent on the mathematical framework. These manipulations are relatively trivial when finite geometry or purely algebraic objects are involved. They may present serious difficulties in a functional-analysis and group-representation context [11]. For the specific passage (anti-) de Sitter–Poincaré we know the following scheme concerning the relationships between *massive* representations,

$$\begin{array}{ccccc} & & \mathcal{P}^{>}(m,s) & \xleftarrow[\kappa\to 0]{E_0\to\infty} & \mathcal{D}^{>}_{+}(E_0,s) \\ \mathcal{D}_{-}(E_0,S) & \xrightarrow[\kappa\to 0]{E_0\to\infty} & \oplus & & \\ & & \mathcal{P}^{<}(-m,s) & \xleftarrow[\kappa\to 0]{E_0\to\infty} & \mathcal{D}^{<}_{+}(-E_0,s). \end{array} \tag{1.2}$$

$\mathcal{P}^{>}$ and $\mathcal{P}^{<}$ are respectively the positive-energy and the negative-energy Wigner representations. $D_{-}(E_0,s)$ is a principal-series representation of $\mathrm{SO}_0(4,1)$ characterized by a spin s and a parameter E_0 associated with the unitary character of the non-compact time-translation subgroup $\mathrm{SO}_0(1,1)$ (see figure 2(*a*)). $\mathcal{D}^{>}(E_0,s)$ and $\mathcal{D}^{<}(-E_0,s)$ are respectively minimal-weight and maximal-weight representations of $\mathrm{SO}_0(3,2)$ that belong to the discrete series for $E_0 > s+2$. E_0 is the positive lower bound of the discrete spectrum of the compact time-translation generator corresponding to a subgroup $\mathrm{SO}(2)$ (see figure 2(*b*)). Contractions in (1.2) are performed by keeping the product $E_0\xi$ equal to one. It is clear from (1.2) that the $\mathrm{SO}_0(4,1)$ relativity ignores the sign of the energy whereas the $\mathrm{SO}_0(3,2)$ relativity and its Poincaré limit distinguish it. This is the first reason why we favour the anti-de Sitter kinematics, even though there exists [12] a sort of selection rule, based on the existence of a *causality semi-group* in $\mathrm{SO}_0(4,1)$, that allows one to extract from $\mathcal{D}_{-}(E_0,s)$ only what contracts onto $\mathcal{P}^{>}(m,s)$. Our second motivation rests upon the opportunity of exploiting very rich analytic structures that appear in the $\mathrm{SO}_0(3,2)$ geometry. The *phase space* which is, for a given relativity, the homogeneous space

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is in this precise case $\mathrm{SO}_0(3,2)/\mathrm{SO}(3)\times\mathrm{SO}(2)$, i.e. a classical domain in the Cartan terminology [13]. The existence of a discrete series of representations jointly to that of an analytic (*Fock–Bargmann*) carrier space [14] deserves a careful investigation. A part of this programme consists in studying what persists and what disappears among those analytic structures after contraction onto the Poincaré group and this is the content of the present paper.

It seems a general trend of contemporary physics to deal with phase space rather than with spacetime as a natural arena for describing physical processes. The reason is clear in classical mechanics and in statistical physics since points in the phase space of a system immediately represent its physical states. Quantization is naturally and historically based on the phase space Hamiltonian formalism, although physicists are still looking for or making proposals [15] for non-equivocal criteria for defining quantum observables from their classical counterparts. Difficulties mainly arise in the spacetime formulation of the Poincaré case: localizability is in conflict with causality and some inconsistencies arise in theories for high-spin interacting systems [16]. They might be eluded in $\mathrm{SO}_0(3,2)$-quantum mechanics by exploiting at once the regularization due to the curvature and the Fock–Bargmann analytic structure. The latter emerges from a Kählerian phase space associated to any massive elementary system.

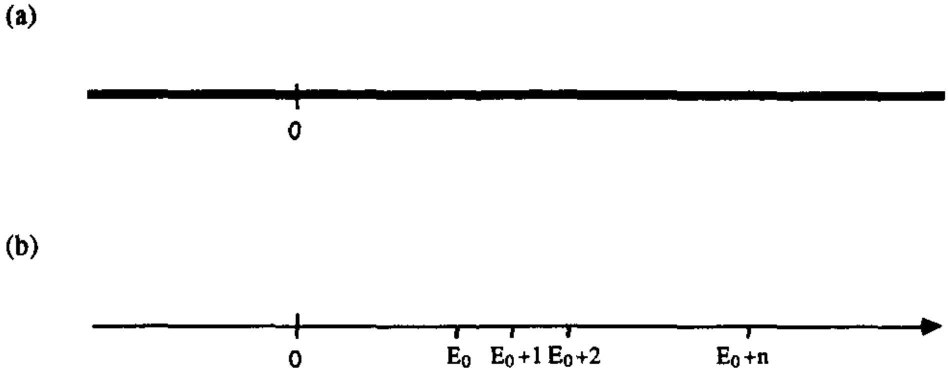

Figure 2. (*a*) Spectrum of the $\mathrm{SO}_0(1,1)$ time-translation generator in the $\mathcal{D}_-(E_0,s)$ representation of $\mathrm{SO}_0(4,1)$. (*b*) Spectrum of the $\mathrm{SO}(2)$ time-translation generator in the $\mathcal{D}^{>}_{+}(E_0,s)$ representation of $\mathrm{SO}_0(3,2)$.

Then a quantum theory on phase space [16] with nice attributes like localization can be developed without any use of intricate *spacetime* $\leftrightarrow$ *phase space* transformation (such as Weyl transformation). Curiously enough one recovers the same features as those existing for the Weyl–Heisenberg group where an elementary system, namely the harmonic oscillator, has the complex plane as phase space. Setting in a comprehensive and consistent way a quantum mechanics on the $\mathrm{SO}_0(3,2)$-phase space for free and interacting systems and examining the contraction limits of the theory is our eventual goal [17, 18].

The organization is as follows. We begin (section 2) by a short presentation of the group $\mathrm{SO}_0(3,2)$ as acting on the anti-de Sitter spacetime. The language of complex quaternions appeared to us as the most adapted for describing the special isomorphism $\mathrm{SO}_0(3,2) \cong \mathrm{Sp}(4,\mathbf{R})/Z_2$, and the classical domain $\mathcal{D}^{(3)} \cong \mathrm{SO}_0(3,2)/\mathrm{SO}(3) \times \mathrm{SO}(2)$ (section 3). In section 4, we make explicit the Fock–Bargmann Hilbert space of analytic functions on $\mathcal{D}^{(3)}$ and the corresponding discrete-series representation. The contraction procedure is then carried out at the finite-dimensional level (section 5), i.e. at the level of the spacetime, group and phase space. It is finally achieved in section 6 at the level of the generators of the Fock–Bargmann representation: we show how to recover the Wigner representation at the cost of losing the original analyticity and square-integrability properties (*polarization conditions*). All this is performed in a heuristic way, and we refer to [17, 18] for mathematical precisions. In [18] geometric quantization of the classical Kählerian structure has been worked out in a comprehensive manner and allows one to solve the two problems emerging from the present approach. In [18], the non-natural rescaling used here in equation (6.8) and the singular terms appearing in the contraction limit of the $\mathrm{SO}_0(3,2)$ infinitesimal generators of the holomorphic UIR (6.7), are avoided by performing the contraction at the prequantized level. However the contraction at the quantum level as it is carried out in the present paper is an illustration of a non-immediate link between the notion of polarization and the contraction of representations. This is clearly established since the singular part of the contracted generators are just the Poincaré polarization corresponding to the holomorpic polarization of $\mathrm{SO}_0(3,2)$.

Previous work on the anti-de Sitter–Poincaré contraction procedure for the two-dimensional case can be found in [19].

2. Quaternionic realization of the anti-de Sitter kinematical group

Anti-de Sitter spacetime is characterized by a constant curvature $\kappa > 0$. It can be vizualized as the unit pseudo-sphere in $\mathbf{R}^{2+3}$,

$$y^2 \equiv \delta_{\alpha\beta} y^\alpha y^\beta \equiv y_5^2 + y_0^2 - y_1^2 - y_2^2 - y_3^2 = \kappa^{-2} \tag{2.1}$$

where $\delta_{\alpha\beta} = \mathrm{diag}(+1,+1,-1,-1,-1)$ and the indices $\alpha, \beta, \ldots$ run on the values $\{5,0,1,2,3\}$. We shall reserve the Minkowski indices $\mu, \nu, \ldots$ to the subset $\{0,1,2,3\}$ and the spatial indices $i, j, \ldots$ to the subset $\{1,2,3\}$†.

The one-sheeted hyperboloid (2.1) can be given global coordinates $\{x^\mu\}$ which are tensorial with respect to the Minkowski metric $\delta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$,

$$\begin{aligned} y_5 &= Y \cos \kappa x_0 \\ y_0 &= Y \sin \kappa x_0 \\ y_i &= x_i \end{aligned} \tag{2.2a}$$

where

$$Y = \left(\kappa^{-2} + \boldsymbol{x}^2\right)^{1/2} \qquad x_i \in \mathbf{R} \qquad \text{and} \qquad -\pi \leqslant \kappa x_0 < \pi . \tag{2.2b}$$

We refer to Fronsdal [10] for the various geometrical and physical properties of (2.1).

Let us denote by $X_{\alpha\beta}$ the infinitesimal generator associated with the (pseudo) rotation in the (α, β) plane. In terms of the 5×5 matrix representation of $\mathrm{SO}_0(3,2)$ they are given by

$$\left(X_{\alpha\beta}\right)_\gamma{}^\sigma = \delta_\alpha{}^\sigma \delta_{\beta\gamma} - \delta_{\alpha\gamma} \delta_\beta{}^\sigma . \tag{2.3a}$$

They satisfy the commutation rules

$$[X_{\alpha\beta}, X_{\alpha\gamma}] = \begin{cases} \delta_{\alpha\alpha} X_{\beta\gamma} & \text{two equal indices} \\ 0 & \text{all indices are different.} \end{cases} \tag{2.3b}$$

In order to display the homomorphism between $\mathrm{SO}_0(3,2)$ and $\mathrm{Sp}(4,\mathbf{R})$ [13], we associate to any 5-uple (y^α) in $\mathbf{R}^{2+3}$ the following 4×4 matrix with complex entries,

$$\Gamma(y) = y^\alpha \Gamma_\alpha \tag{2.4a}$$

where the five matrices Γ_α are given by

$$\begin{aligned} &\Gamma_0 = \begin{pmatrix} \mathrm{i}1_2 & 0_2 \\ 0_2 & -\mathrm{i}1_2 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} 0_2 & \mathrm{i}\sigma_1 \\ -\mathrm{i}\sigma_1 & 0_2 \end{pmatrix} \\ &\Gamma_2 = \begin{pmatrix} 0_2 & -\mathrm{i}\sigma_2 \\ \mathrm{i}\sigma_2 & 0_2 \end{pmatrix} \qquad \Gamma_3 = \begin{pmatrix} 0_2 & \mathrm{i}\sigma_3 \\ -\mathrm{i}\sigma_3 & 0_2 \end{pmatrix} \\ &\Gamma_5 = 1_4 \end{aligned} \tag{2.4b}$$

† The missing index value 4 is taken apart in view of quaternionic use. See later.

where the σ_is are the usual Pauli matrices. Hence the general expression of $\Gamma(y)$ is

$$\Gamma(y) = \begin{pmatrix} (y_+)1_2 & \boldsymbol{Y} \\ -\boldsymbol{Y} & (y_-)1_2 \end{pmatrix} \tag{2.5a}$$

where

$$y_\pm = y_5 \pm \mathrm{i}y_0 \qquad \text{and} \qquad \boldsymbol{Y} \equiv \begin{pmatrix} \mathrm{i}y^3 & \mathrm{i}y^1 - y^2 \\ \mathrm{i}y^1 + y^2 & -\mathrm{i}y^3 \end{pmatrix}. \tag{2.5b}$$

We note that $\det \Gamma(y) = y_\alpha y^\alpha$.

Exploiting the well known isomorphism between the algebra $\mathcal{M}_2(\mathbf{C})$ of 2×2 complex matrices and that one of complex quaternions, $\Gamma(y)$ can be written as a 2×2 complex quaternionic matrix. Actually this isomorphism associates with $\boldsymbol{Y}$ in (2.5*b*), the complex pure-vector quaternion $\boldsymbol{y}$ and to, respectively, $(y_+)1_2$ and $(y_-)1_2$ in (2.5a), the complex scalar quaternions y_+ and y_-; this yields

$$\Gamma(y) = \begin{pmatrix} y_+ & \boldsymbol{y} \\ -\boldsymbol{y} & y_- \end{pmatrix}. \tag{2.6}$$

General facts about complex quaternions are now outlined [17]. A complex quaternion z is the scalar–vector pair $(z^4, \boldsymbol{z})$, defined by

$$z = (z^4, \boldsymbol{z}) \equiv z^4 + z^1 e_1 + z^2 e_2 + z^3 e_3 \tag{2.7a}$$

where $(z^1, z^2, z^3, z^4) \in \mathbf{C}^4$ and $\{e_1, e_2, e_3\}$ satisfy the quaternionic algebra

$$\begin{cases} e_i^2 = -1 & \\ e_i e_j = e_k & \text{for } (ijk) \text{ an even permutation of (123).} \end{cases} \tag{2.7b}$$

In the following we shall sometimes designate z^4 by $(z)_s$ (s for scalar), and we shall also make use of the decomposition of z into real and imaginary parts (which are real quaternions) i.e. $z = x + \mathrm{i}y$. The *complex conjugate*, the *quaternionic conjugate* and the *adjoint* of z are respectively defined through

$$\bar{z} = (\bar{z}^4, \bar{\boldsymbol{z}}) \qquad \tilde{z} = (z^4, -\boldsymbol{z}) \qquad \text{and} \qquad z^* = \bar{\tilde{z}} = \tilde{\bar{z}} \tag{2.8a}$$

or equivalently

$$\bar{z} = x - \mathrm{i}y \qquad \tilde{z} = \tilde{x} + \mathrm{i}\tilde{y} \qquad \text{and} \qquad z^* = \tilde{x} - \mathrm{i}\tilde{y}. \tag{2.8b}$$

The product of two complex quaternions is

$$zz' = (z^4, \boldsymbol{z})(z'^4, \boldsymbol{z}') = (z^4 z'^4 - \boldsymbol{z} \cdot \boldsymbol{z}', z^4 \boldsymbol{z}' + z'^4 \boldsymbol{z} + \boldsymbol{z} \times \boldsymbol{z}') \tag{2.9}$$

where $\boldsymbol{z} \cdot \boldsymbol{z}'$ and $\boldsymbol{z} \times \boldsymbol{z}'$ are the analytic continuations of respectively the dot product and the cross product in $\mathbf{R}^3$. The *determinant* of a complex quaternion z is defined by

$$\det z = \det \tilde{z} = z\tilde{z} = \tilde{z}z = (z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 = \|x\|^2 - \|y\|^2 + 2\mathrm{i}x \cdot y \tag{2.10a}$$

where $\|x\|^2 = x \cdot x$ and '$\cdot$' is the Euclidian dot product in $\mathbf{R}^4$. It is equal to $\det Z$, where Z is the $\mathcal{M}_2(\mathbf{C})$ counterpart of z:

$$Z = \begin{pmatrix} z^4 + \mathrm{i}z^3 & \mathrm{i}z^1 - z^2 \\ \mathrm{i}z^1 + z^2 & z^4 - \mathrm{i}z^3 \end{pmatrix}. \tag{2.10b}$$

From (2.10) one derives the expression of the inverse z^{-1} of z, which exists whenever $\det z$ is not zero,

$$z^{-1} = \frac{\tilde{z}}{\det z}. \tag{2.11}$$

In terms of complex quaternionic algebra, elements of $\mathrm{Sp}(4,\mathbf{R})$ are 2×2 complex quaternionic matrices of the form [13, 20],

$$\mathrm{Sp}(4,\mathbf{R}) \ni g \equiv \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \tag{2.12a}$$

such that the inverse g^{-1} of g is given by

$$g^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{g}^{\mathrm{t}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a^* & \tilde{b} \\ -b^* & \tilde{a} \end{pmatrix}. \tag{2.12b}$$

This means that the complex quaternionic entries have to obey,

$$aa^* - bb^* = 1 \qquad \text{and} \qquad a\tilde{b} = -b\tilde{a} \tag{2.13a}$$

or equivalently

$$a^*a - \tilde{b}\bar{b} = 1 \qquad \text{and} \qquad a^*b = -\tilde{b}\bar{a}. \tag{2.13b}$$

$\mathrm{Sp}(4,\mathbf{R})$ acts on the matrices $\Gamma(y)$ in the following way,

$$\mathrm{Sp}(4,\mathbf{R}) \ni g : \Gamma(y) \longmapsto \Gamma(y') = g\Gamma(y)\tilde{g}^{\mathrm{t}} \tag{2.14}$$

or more explicitly as

$$y'_+ = y_+ \det a + y_- \det b + 2(ay\tilde{b})_{\mathrm{s}} \tag{2.15a}$$

$$\boldsymbol{y}' = -y_+ ab^* + y_- ba^* + a\boldsymbol{y}a^* + b\boldsymbol{y}b^*. \tag{2.15b}$$

Thus the homomorphim $\mathrm{Sp}(4,\mathbf{R}) \to \mathrm{SO}_0(3,2)$ is easily deduced from the above

$$\mathrm{Sp}(4,\mathbf{R}) \ni g \longmapsto R_g \in \mathrm{SO}_0(3,2) \tag{2.16a}$$

$$g\Gamma(y)\tilde{g}^{\mathrm{t}} = \Gamma(y') = \Gamma(R_g y). \tag{2.16b}$$

The matrix R_g is realized as the 5×5 block matrix,

$$R_g = \begin{array}{c} \\ \binom{5}{0} \\ \left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right) \end{array} \begin{array}{c} \begin{array}{cc} (5\ 0) & (1\ 2\ 3) \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \tag{2.17a}$$

where the blocks are given by

$$A = \begin{pmatrix} \mathrm{Re}(\det a + \det b) & -\mathrm{Im}(\det a - \det b) \\ \mathrm{Im}(\det a + \det b) & \mathrm{Re}(\det a - \det b) \end{pmatrix} \tag{2.17b}$$

$$B = \begin{pmatrix} m^1 & m^2 & m^3 \\ n^1 & n^2 & n^3 \end{pmatrix} \qquad C^{\mathrm{t}} = \begin{pmatrix} m'^1 & m'^2 & m'^3 \\ n'^1 & n'^2 & n'^3 \end{pmatrix}. \tag{2.17c}$$

Here the real pure-vector quaternions $\boldsymbol{m}$ and $\boldsymbol{n}$ (resp. $\boldsymbol{m}'$ and $\boldsymbol{n}'$) are the real and the imaginary parts of the complex pure vector quaternion $2\tilde{a}b$ (resp. $2ba^*$). The columns of the 3×3 matrix D are made up with the components of three real pure-vectors,

$$D = (\boldsymbol{p}_{(1)}, \boldsymbol{p}_{(2)}, \boldsymbol{p}_{(3)}) \qquad \text{with} \qquad \boldsymbol{p}_{(i)} = ae_i a^* + be_i b^*. \tag{2.17d}$$

To summarize, we rewrite R_g in the more compact form:

$$R_g = \begin{pmatrix} (\nu_+, \mathrm{i}\nu_-) & (\boldsymbol{m}, \boldsymbol{n})^{\mathrm{t}} \\ (\boldsymbol{m}', \boldsymbol{n}') & (\boldsymbol{p}_{(1)}, \ \boldsymbol{p}_{(2)}, \ \boldsymbol{p}_{(3)}) \end{pmatrix} \tag{2.18}$$

where the two complex numbers $\nu_\pm \equiv \det a \pm \det b$ and the seven real pure-vector quaternions $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{m}', \boldsymbol{n}'$ and $\boldsymbol{p}_{(i)}$ have to be written as column vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ respectively. Note also the relationships between the R_g blocks due to the invariance of the metric in $\mathbf{R}^{2+3}$,

$$R_g^{\mathrm{t}} \Delta_{2,3} R_g = \Delta_{2,3} \qquad \Delta_{2,3} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_3 \end{pmatrix} \tag{2.19a}$$

$$A^{\mathrm{t}} A - C^{\mathrm{t}} C = 1_2, \quad D^{\mathrm{t}} D - B^{\mathrm{t}} B = 1_3 \qquad \text{and} \qquad A^{\mathrm{t}} B = C^{\mathrm{t}} D \tag{2.19b}$$

or equivalently,

$$AA^{\mathrm{t}} - BB^{\mathrm{t}} = 1_2 \qquad DD^{\mathrm{t}} - CC^{\mathrm{t}} = 1_3 \qquad \text{and} \qquad AC^{\mathrm{t}} = BD^{\mathrm{t}}. \tag{2.19c}$$

3. A classical domain as anti-de Sitterian phase space

As any simple Lie group, $\mathrm{Sp}(4, \mathbf{R})$ admits a Cartan factorization [21],

$$\mathrm{Sp}(4, \mathbf{R}) = PK \Longleftrightarrow \mathrm{Sp}(4, \mathbf{R}) \ni g = pk \tag{3.1}$$

where $K = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$ is the maximal compact subgroup,

$$K \ni k = \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} \qquad u = \mathrm{e}^{\mathrm{i}\theta/2}\xi \qquad 0 \leqslant \theta \leqslant 2\pi \tag{3.2}$$

with ξ a real quaternion of modulus 1, i.e. an element of $\mathrm{SU}(2)$. The Cartan factorization (3.1) is associated with the existence of the involution [21],

$$g \longmapsto \sigma(g) = [(g^{\mathrm{t}})^*]^{-1} = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix} \tag{3.3a}$$

such that

$$\sigma(k) = k \qquad \text{and} \qquad \sigma(p) = p^{-1}. \tag{3.3b}$$

This implies

$$p = (gg^{+})^{1/2}. \tag{3.4}$$

Using the fact that any quaternion a can be written in the polar form

$$a = (aa^{*})^{1/2}u \qquad uu^{*} = u^{*}u = 1 \tag{3.5}$$

we perform the above-mentioned factorization for the quaternionic representation of $\mathrm{Sp}(4, \mathbf{R})$ presented in (2.12), thus

$$g = \begin{pmatrix} (aa^{*})^{1/2}u & b \\ -\bar{b} & (\bar{a}\tilde{a})^{1/2}\bar{u} \end{pmatrix} = \begin{pmatrix} (aa^{*})^{1/2} & b\tilde{u} \\ -\bar{b}u^{*} & (\bar{a}\tilde{a})^{1/2} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}. \tag{3.6}$$

From (2.13b) we obtain (a being always invertible as a consequence of (2.13a) i.e. $\det a \neq 0$)

$$\begin{aligned}(aa^{*})^{-1} &= 1 - (\bar{b}a^{-1})^{*}(\bar{b}a^{-1}) \\ &= 1 - (b\bar{a}^{-1})(b\bar{a}^{-1})^{*}\end{aligned} \tag{3.7}$$

where $b\bar{a}^{-1}$ is easily seen to be a pure-vector quaternion and therefore it will be denoted

$$\boldsymbol{z} \equiv b\bar{a}^{-1}. \tag{3.8}$$

Using (3.6), (3.7) and (3.8), the factor p in (3.1) takes the following $\boldsymbol{z}$-dependent form,

$$p(\boldsymbol{z}) = \begin{pmatrix} (1 + \boldsymbol{z}\bar{\boldsymbol{z}})^{-1/2} & \boldsymbol{z}(1 + \bar{\boldsymbol{z}}\boldsymbol{z})^{-1/2} \\ -\bar{\boldsymbol{z}}(1 + \boldsymbol{z}\bar{\boldsymbol{z}})^{-1/2} & (1 + \bar{\boldsymbol{z}}\boldsymbol{z})^{-1/2} \end{pmatrix}. \tag{3.9}$$

Actually, $\boldsymbol{z}$ is confined to lie in a bounded domain of $\mathbf{C}^3$, indeed

$$(1 + \boldsymbol{z}\bar{\boldsymbol{z}}) = (aa^{*})^{-1} > 0 \tag{3.10a}$$

i.e. the $\mathcal{M}_2(\mathbf{C})$ counterpart Z of $\boldsymbol{z}$ obeys

$$1_2 - ZZ^{+} > 0. \tag{3.10b}$$

This means that the spectral radius or the largest eigenvalue of the Hermitian matrix ZZ^{+} is smaller than 1, and this can be formulated as follows

$$\|\boldsymbol{z}\|^2 + \|\boldsymbol{z} \times \bar{\boldsymbol{z}}\| < 1. \tag{3.10c}$$

The domain $\mathcal{D}^{(3)}$ of $\mathbf{C}^3$ described by (3.10a) is an *irreducible bounded symmetric hermitian domain* or a classical domain [13, 22, 23]. Its Shilov boundary is diffeomorphic to $S^1 \times_{Z_2} S^2$, where $\times_{Z_2}$ is the Cartesian product modulo a ± 1-factor.

The $SO_0(3,2)$ counterpart of the factorization (3.1) is easily found using the homomorphism (2.16). Thus, the 5×5 real matrix χ which corresponds to k reads in the formulation (2.18)

$$\chi = \begin{pmatrix} (e^{i\theta}, ie^{i\theta}) & (o, o)^t \\ (o, o) & (\xi e_1\tilde{\xi}, \xi e_2\tilde{\xi}, \xi e_3\tilde{\xi}) \end{pmatrix} \tag{3.11}$$

where θ and ξ are those introduced in (3.2). We easily recognize the anti-de Sitterian-time translation subgroup $SO(2)$ and the 3-space rotation subgroup $SO(3)$ associated respectively to the parameter θ and to the $SU(2)$ quaternion ξ. Along the same way, the 5×5 real matrix Π which corresponds to p reads in the formulation (2.17a)

$$\Pi = \Pi(X) = \begin{pmatrix} (1_2 + XX^t)^{1/2} & X \\ X^t & (1_3 + X^tX)^{1/2} \end{pmatrix} \tag{3.12a}$$

where the 2×3 matrix X is given, according to the notation in (2.17c) and (2.18), by

$$X = \begin{pmatrix} \alpha^1 & \alpha^2 & \alpha^3 \\ \beta^1 & \beta^2 & \beta^3 \end{pmatrix} = \left(\frac{\zeta+\bar{\zeta}}{2}, \frac{\zeta-\bar{\zeta}}{2i}\right)^t \tag{3.12b}$$

with

$$\begin{aligned} \zeta = \alpha + i\beta &\equiv 2z\,[\det(1+z\bar{z})]^{-1/2} \\ &= 2z\left[1 - 2z\cdot\bar{z} + |z\cdot z|^2\right]^{-1/2}. \end{aligned} \tag{3.12c}$$

Note that $\Pi^t = \Pi$ whereas $\chi^t = \chi^{-1}$; these are the $SO_0(3,2)$ counterparts of (3.3b), where the Cartan involution takes the form $\sigma(R) = (R^t)^{-1}$. Also, the square roots that appear in (3.12a) are a simple consequence of (2.19b) and (2.19c) when A and D are symmetrical and $C = B^t$. The latter properties are clearly put in evidence from (2.17b), (2.17c) and (2.17d) when a and b are those of (3.9). Actually, A and D are

$$A = \begin{pmatrix} \dfrac{1+\mathrm{Re}\det z}{[\det(1+z\bar{z})]^{1/2}} & \dfrac{\mathrm{Im}m\det z}{[\det(1+z\bar{z})]^{1/2}} \\ \dfrac{\mathrm{Im}m\det z}{[\det(1+z\bar{z})]^{1/2}} & \dfrac{1-\mathrm{Re}\det z}{[\det(1+z\bar{z})]^{1/2}} \end{pmatrix} \tag{3.13a}$$

$$D = \left(\delta(e_1 - ze_1\bar{z})\delta \qquad \delta(e_2 - ze_2\bar{z})\delta \qquad \delta(e_3 - ze_3\bar{z})\delta\right). \tag{3.13b}$$

Here $\delta = (1+z\bar{z})^{-1/2}$ and the property

$$z(1+\bar{z}z)^{-1/2} = (1+z\bar{z})^{-1/2}z \tag{3.13c}$$

has been used in a systematic way in the computations. Inverting (3.12c) yields

$$\begin{aligned} z &= \zeta\left[2 + \|\zeta\|^2 + \left((2+\|\zeta\|^2)^2 - |\det\zeta|^2\right)^{1/2}\right]^{-1/2} \\ &= \frac{\zeta}{\sqrt{2}}\left[\left(2+\|\zeta\|^2+|\det\zeta|\right)^{1/2} - \left(2+\|\zeta\|^2-|\det\zeta|\right)^{1/2}\right]. \end{aligned} \tag{3.14}$$

Clearly ζ may run throughout the whole space $\mathbf{C}^3$, whereas z is confined to lie in $\mathcal{D}^{(3)}$.

The two coset spaces $\mathrm{Sp}(4,\mathbf{R})/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))$ and $\mathrm{SO}_0(3,2)/\mathrm{SO}(3)\times\mathrm{SO}(2)$ are homogeneous spaces for the left action of $\mathrm{Sp}(4,\mathbf{R})$ and $\mathrm{SO}_0(3,2)$ respectively. We define the transformations of $\mathcal{D}^{(3)}$ and $\mathbf{C}^3$ through the decompositions,

$$\mathrm{Sp}(4,\mathbf{R}) \ni g : p(z) \longmapsto gp(z) = p(g\cdot z)k \tag{3.15a}$$

$$\mathrm{SO}_0(3,2) \ni R : \Pi(X) \longmapsto R\Pi(X) = \Pi(R\cdot X)\chi. \tag{3.15b}$$

The result is for the first one a generalized *homographic* action, namely

$$z' \equiv g\cdot z = \big(az+b\big)\big(-\bar{b}z+\bar{a}\big)^{-1} \tag{3.16a}$$

$$= \big(zb^* + a^*\big)\big(z\tilde{a} - \tilde{b}\big). \tag{3.16b}$$

We end this section by displaying some properties of the classical domain $\mathcal{D}^{(3)}$. $\mathcal{D}^{(3)}$ is Kählerian [13, 22, 23] and it has G-invariant metric and G-invariant 2-form with respect to the analytic diffeomorphism (3.16). Both arise from the Kählerian potential $\ln K(z,\bar{z})$, where $K(z,\bar{z})$ is the Bergman kernel

$$K(z,\bar{z}) = \frac{1}{V}\,[\det(1+z\bar{z})]^{-3}. \tag{3.17}$$

The coefficient V is the Euclidian volume in $\mathcal{D}^{(3)}$

$$V = \int_{\mathcal{D}^{(3)}} \dot{z} = \frac{\pi^3}{24} \tag{3.18}$$

where $\dot{z} = \mathrm{d}^3x\,\mathrm{d}^3y$ and $z = x + \mathrm{i}y$. Therefore the Bergman kernel yields the Riemannian metric $h_{i\bar{j}}$ on $\mathcal{D}^{(3)}$, given by

$$\begin{aligned} \mathrm{d}s^2 = -h_{i\bar{j}}\mathrm{d}z^i d\bar{z}^j &= \mathrm{d}\bar{\mathrm{d}}\ln K(z,\bar{z}) \\ &= -\frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^j}\ln K(z,\bar{z})\mathrm{d}z^i\mathrm{d}\bar{z}^j. \end{aligned} \tag{3.19}$$

Here, the barred indices correspond to the complex conjugate variables $(z^{\bar{i}} \equiv \bar{z}^i)$ as usual and the minus sign is necessary since the spatial part δ_{ij} of the anti-de Sitterian metric is negative. The Bergman kernel also yields the closed 2-form

$$\omega = -\frac{\mathrm{i}}{2}h_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \tag{3.20}$$

with $\mathrm{d}\omega = 0$. Because its symplectic structure, $\mathcal{D}^{(3)}$ will be called phase space for the double covering $\mathrm{Sp}(4,\mathbf{R})$ of the kinematical group $\mathrm{SO}_0(3,2)$ of anti-de Sitter space. Finally, we give the explicit form of the invariant measure on $\mathcal{D}^{(3)}$ with respect to (3.16):

$$\mathrm{d}\mu(z) = \frac{1}{V}\,[\det(1+z\bar{z})]^{-3}\,\dot{z}. \tag{3.21}$$

4. Fock–Bargmann Hilbert space and discrete series representations of Sp(4,R)

We come now to the description of the discrete series representations of $\mathrm{Sp}(4,\mathbf{R})$ (actually of its universal covering $\overline{\mathrm{Sp}(4,\mathbf{R})}$). Let us first describe the representation space. We denote by $\mathcal{F}^{(E_0,s)}$ the space of holomorphic $(2s+1)$-vector functions

$$\mathcal{D}^{(3)} \ni z \longmapsto f(z) \in \mathbf{C}^{2s+1} \tag{4.1}$$

that are square integrable with respect to the bilinear form

$$\begin{aligned}(f_1, f_2)_{(E_0,s)} = \mathcal{N}(E_0, s) \int_{\mathcal{D}^{(3)}} (f_1(z))^+ D^s\left(\frac{1}{1+z\bar{z}}\right)\\ \times f_2(z)\,[\det(1 + z\bar{z})]^{E_0+s-3}\,\dot{z}.\end{aligned} \tag{4.2}$$

D^s denotes here the irreducible $(2s+1)\times(2s+1)$-matrix representation of $\mathrm{SU}(2)$, holomorphically extended to $\mathcal{M}_2(\mathbf{C})$. The constant $\mathcal{N}(E_0, s)$ is chosen in such a way that the particular function $f(z) = (f(z))_{q-s\leqslant q\leqslant s}$ defined by

$$\big(f(z)\big)_q = \delta_{sq} \tag{4.3}$$

have norm one. Explicitly,

$$\mathcal{N}(E_0, s) = \left(\frac{2}{\pi}\right)^3 \left(E_0 + s - \frac{3}{2}\right)(E_0 - s - 1)(E_0 - s - 2). \tag{4.4}$$

Therefore, the Hilbert space $\mathcal{F}^{(E_0,s)}$ is non-trivial if and only if $E_0 > s + 2$. Modifications of the form are required in order to relax this condition until reaching the unitarity lowest limits [24]:

$$\begin{array}{lll} E_0 = s+1 & s \geqslant 1 & \text{massless} \\ E_0 = 1 & s = 1/2 & \text{singleton Di} \\ E_0 = 1/2 & s = 0 & \text{singleton Rac.} \end{array} \tag{4.5}$$

Here the correct definition of the scalar product involves the lower dimensional manifold S^2 embedded in the Shilov boundary $S^1 \times S^2$ (See for instance Onofri [25] for a definition of $\mathcal{F}^{(E_0,s=0)}$ when $\frac{1}{2} \leqslant E_0 \leqslant 2$). The positive parameter E_0 is considered as a minimal energy for reasons that will soon appear.

We now define the representation operator $T^{(E_0,s)}(g)$,

$$\big(T^{(E_0,s)}(g)f\big)(z) = \big[\det(-\bar{b}z + \bar{a})\big]^{-E_0-s} D^s(zb^* + a^*) f(g^{-1}\cdot z) \tag{4.6}$$

for $f \in \mathcal{F}^{(E_0,s)}$ and

$$g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}(4,\mathbf{R}).$$

The representatives of the Lie algebra elements are then given by the unitary operators

$$L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta} \tag{4.7a}$$

where $M_{\alpha\beta}$ is the *orbital* part and $S_{\alpha\beta}$ is the *spin* part. They satisfy the following commutation rules,

$$[L_{\alpha\beta}, L_{\gamma\rho}] = \mathrm{i}(\delta_{\alpha\gamma} L_{\beta\rho} + \delta_{\beta\rho} L_{\alpha\gamma} - \delta_{\alpha\rho} L_{\beta\gamma} - \delta_{\beta\gamma} L_{\alpha\rho}). \tag{4.7b}$$

Explicitly, we obtain for the orbital part

$$M_{50} = \boldsymbol{z}\cdot\boldsymbol{\nabla}_z + E_0 \qquad \boldsymbol{\nabla}_z = \left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^3}\right) \tag{4.8a}$$

$$M_{ij} = \mathrm{i}\left(z^i \frac{\partial}{\partial z^j} - z^j \frac{\partial}{\partial z^i}\right) \tag{4.8b}$$

$$M_{5i} = \mathrm{i}\left(\frac{1 + \boldsymbol{z}\cdot\boldsymbol{z}}{2}\frac{\partial}{\partial z^i} - z^i(E_0 + \boldsymbol{z}\cdot\boldsymbol{\nabla}_z)\right) \tag{4.8c}$$

$$M_{0i} = -\left(\frac{1 - \boldsymbol{z}\cdot\boldsymbol{z}}{2}\frac{\partial}{\partial z^i} + z^i(E_0 + \boldsymbol{z}\cdot\boldsymbol{\nabla}_z)\right). \tag{4.8d}$$

The spin parts are given in the appendix. It can be checked that the second-order Casimir operator

$$C_2 = \tfrac{1}{2} L_{\alpha\beta} L^{\alpha\beta} \tag{4.9a}$$

takes on the value

$$C_2 = [E_0(E_0 - 3) + s(s+1)]\,\mathrm{Id} \tag{4.9b}$$

identically on $\mathcal{F}^{(E_0,s)}$. Moreover, $L_{50} = M_{50}$ is the ladder operator or *anti-de Sitterian energy* operator whose eigenvalues on the representation space $\mathcal{F}^{(E_0,s)}$ are $E_0, E_0+1, \ldots, E_0+n, \ldots$. We are here in presence of lowest-weight representations. The lowest-weight state $f_0(\boldsymbol{z})$ is defined by (4.3).

The Fock–Bargmann spaces $\mathcal{F}^{(E_0,s)}$ are reproducing-kernel spaces. The $(2s+1)\times(2s+1)$ matrix-valued reproducing-kernel is given by

$$K(\boldsymbol{z}, \boldsymbol{z}') = [\det(1 + \boldsymbol{z}\boldsymbol{z}')]^{-E_0-s}\, D^s(1 + \boldsymbol{z}\bar{\boldsymbol{z}}'). \tag{4.10}$$

We recall its reproducing property

$$f(\boldsymbol{z}) = \left(K^+(\boldsymbol{z}, .), f\right)_{(E_0,s)}. \tag{4.11}$$

An expansion of K allows one to find an orthonormal basis for $\mathcal{F}^{(E_0,s)}$.

5. Anti-de Sitter–Poincaré contraction: geometrical aspects

In anti-de Sitterian physics, there exists an universal length, namely κ^{-1}, to which any other length-like physical quantity has to be compared (e.g. the global coordinates, (x^0, x^i) in (2.2)). Nevertheless, as long as we deal with a non-zero curvature, all the parameters of the kinematical transformations of anti-de Sitter space are (pseudo-) angles associated with (pseudo-) rotations of the simple group $\mathrm{SO}_0(3,2)$. But if we have in mind the *flatland* limit $\kappa \to 0$ or if we examine things from what we believe to be our natural arena, namely the Minkowski space, some of those parameters are length-like. Actually, the four ones which are almost the components of a four-vector translation in Poincaré kinematics are length-like. They correspond to the four $SO_0(3,2)$ generators $X_{5\mu}$ given in (2.3), and they are introduced after rescaling the corresponding dimensionless pseudo-angles $\Theta^{5\mu}$,

$$q^{\mu} = \kappa^{-1}\Theta^{5\mu}. \tag{5.1}$$

The remnant parameters do not need such a rescaling since they correspond to the unmodified Lorentz subgroup $SO_0(3,1)$.

If we examine the behaviour at $\kappa \simeq 0$ of the y-coordinates (2.2) for the hyperboloid (2.1),

$$y_5 \simeq \kappa^{-1} \tag{5.2}$$
$$y_0 = x_0 + o(\kappa) \tag{5.3}$$
$$y_i = x_i \tag{5.4}$$

we see clearly how to rescale an arbitrary $SO_0(3,2)$-action in order to give it a physical meaning at the limit $\kappa \to 0$. Since only the first component (5.2) becomes singular, this rescaling corresponds to the similitude matrix [2]:

$$R \in SO_0(3,2) \longmapsto \Delta(\kappa)R\Delta^{-1}(\kappa) \tag{5.5}$$

where

$$\Delta(\kappa) = \begin{pmatrix} \kappa & 0 \\ 0 & 1_4 \end{pmatrix}. \tag{5.6}$$

Any $SO_0(3,2)$-element near the identity can be factorized as follows:

$$R = \left[\prod_{\mu} \exp \Theta_{5\mu} X_{5\mu}\right] \mathcal{L} \tag{5.7}$$

where $\mathcal{L}$ is an element of the orthochronous Lorentz subgroup $SO_0(3,1)$,

$$\mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} \qquad \Lambda \in SO_0(3,1). \tag{5.8}$$

The latter is left-invariant under (5.5) whereas the $(5-\mu)$ pseudo-rotation becomes in the limit

$$\left[\Delta(\kappa)\left[\exp \Theta_{5\mu} X_{5\mu}\right] \Delta^{-1}(\kappa)\right]_{\alpha}{}^{\beta} \longrightarrow \delta_{\alpha}{}^{\beta} + \delta_5{}^{\beta}\delta_{\alpha}{}^{\mu} q_{\mu}. \tag{5.9}$$

The final result on R is the five-dimensional matrix representation of the Poincaré group

$$(q,\Lambda) \equiv \begin{pmatrix} 1 & 0 \\ q & \Lambda \end{pmatrix}. \tag{5.10}$$

We now turn to the zero-curvature limit of the anti-de Sitter phase space $SO_0(3,2)/SO(3) \times SO(2)$. Because we have in mind a Poincaré phase space parametrized by pairs $(\boldsymbol{q}, \boldsymbol{p})$ we adopt the four-momentum description for a Lorentz boost with velocity $\boldsymbol{v}$:

$$L_{\boldsymbol{p}} = \begin{pmatrix} \dfrac{p_0}{mc} & \dfrac{\boldsymbol{p}^{\mathrm{t}}}{mc} \\ \dfrac{\boldsymbol{p}}{mc} & 1_3 + \dfrac{\boldsymbol{p}\boldsymbol{p}^{\mathrm{t}}}{mc(p_0 + mc)} \end{pmatrix} \tag{5.11}$$

where $\boldsymbol{p}/p_0 = -\boldsymbol{v}/c$, and the four-vector $p = (p_0, \boldsymbol{p})$ belongs to the forward mass hyperboloid,

$$\mathcal{V}_m^+ = \{p = (p_0, \boldsymbol{p}) \in \mathbf{R}^4,\ p_0 > 0,\ p^\mu p_\mu = m^2c^2\}\,. \tag{5.12}$$

Then we examine the contraction limit for the rescaled section matrix $\Pi(X)$ given by equations (3.12) and (5.5). It is quite convenient to make explicit its dependence on the real 3-vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (see (3.12)):

$$\Pi(X) = \begin{pmatrix} \dfrac{\rho^2 + \mu}{2\rho} & \dfrac{\nu}{2\rho} & \boldsymbol{\alpha}^{\mathrm{t}} \\ \dfrac{\nu}{2\rho} & \dfrac{\rho^2 - \mu}{2\rho} & \boldsymbol{\beta}^{\mathrm{t}} \\ \boldsymbol{\alpha} & \boldsymbol{\beta} & (1_3 + X^{\mathrm{t}}X)^{1/2} \end{pmatrix} \tag{5.13}$$

where the scalar functions μ, ν, ρ are defined by

$$\mu(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \|\boldsymbol{\alpha}\|^2 - \|\boldsymbol{\beta}\|^2 \tag{5.14a}$$

$$\nu(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 2\boldsymbol{\alpha}\cdot\boldsymbol{\beta} \tag{5.14b}$$

$$\rho(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left[2 + \|\boldsymbol{\alpha}\|^2 + \|\boldsymbol{\beta}\|^2 + 2\left(1 + \|\boldsymbol{\alpha}\|^2 + \|\boldsymbol{\beta}\|^2 + \|\boldsymbol{\alpha}\times\boldsymbol{\beta}\|\right)^{1/2}\right]^{1/2}. \tag{5.14c}$$

Since the limit of $\Delta(\kappa)\Pi(X)\Delta^{-1}(\kappa)$ when $\kappa \to 0$ has the form (5.10), the real part $\boldsymbol{\alpha}$ of $\boldsymbol{\zeta} = \boldsymbol{\alpha} + \mathrm{i}\boldsymbol{\beta}$ must have the asymptotic behaviour

$$\boldsymbol{\alpha} = \kappa\boldsymbol{q} + \mathrm{o}(\kappa^2). \tag{5.15}$$

On the other hand $\boldsymbol{\beta}$ has no reason to vanish at the flat-space limit. Therefore we put

$$\boldsymbol{\beta} = \frac{\boldsymbol{p}}{mc} + \mathrm{o}(\kappa) \tag{5.16}$$

i.e. we define the parameter $\boldsymbol{p}/mc$ as the zeroth-order term of the expansion of $\boldsymbol{\beta}$ in powers of κ. It follows for the rescaled matrix $\Delta(\kappa)\Pi(X)\Delta^{-1}(\kappa)$ the limit matrix at $\kappa = 0$:

$$\begin{aligned} \beta_s(\boldsymbol{q}, \boldsymbol{p}) &\equiv \begin{pmatrix} \begin{pmatrix} 1 \\ \dfrac{\boldsymbol{q}\cdot\boldsymbol{p}}{p_0 + mc} \\ \boldsymbol{q} \end{pmatrix} & \begin{matrix} 0_{1\times 4} \\ L_{\boldsymbol{p}} \end{matrix} \end{pmatrix} \\ &= \left(\left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{p_0 + mc}, \boldsymbol{q}\right), L_{\boldsymbol{p}}\right). \end{aligned} \tag{5.17}$$

Here, we have adopted the notations of [19] to designate the bundle section:

$$\beta_s : \Gamma_l \equiv \mathcal{P}_+^\uparrow(3,1)/\mathrm{SO}(3) \times T \to \mathcal{P}_+^\uparrow(3,1) \tag{5.18}$$

where $T \cong \mathbf{R}$ is the subgroup of time translations. Therefore, the contraction onto the Poincaré group selects a rather remarkable section for the Poincaré phase space,

$$\textit{Poincaré}/(\textit{Spatial Rotations}) \times (\textit{Time Translations}). \tag{5.19}$$

Note that (5.19) is literally the phase space for massive scalar elementary systems, whereas non-zero-spin systems have phase space,

$$\mathcal{P}_+^\uparrow(3,1)/\mathrm{SO}(2) \times T \tag{5.20}$$

which is just the product of Γ_l with the coset $\mathrm{SO}(3)/\mathrm{SO}(2) \cong S^2$. For our present purpose we need not deal with (5.20). The constraint on the $(q_0, \boldsymbol{q}, \boldsymbol{p})$ variables,

$$q_0 = \frac{\boldsymbol{q} \cdot \boldsymbol{p}}{p_0 + mc} \tag{5.21}$$

has very interesting physical implications, previously discussed in [19] and [26]. Let us quote here two of them picked among a set of characteristic ones. First, β_s is the unique section that obeys the equation,

$$[\beta_s(\boldsymbol{q},\boldsymbol{p})]^{-1} = \beta_s(-\boldsymbol{q},-\boldsymbol{p}). \tag{5.22}$$

Secondly, such a section is valid as well for the right coset

$$\Gamma_r = \mathrm{SO}(3) \times T \mid \mathcal{P}_+^\uparrow(3,1). \tag{5.23}$$

The left and right cosets have the same invariant measure

$$\mathrm{d}\mu_s(\boldsymbol{q},\boldsymbol{p}) = \mathrm{d}^3\boldsymbol{q}\frac{\mathrm{d}^3\boldsymbol{p}}{p_0}. \tag{5.24}$$

One could say that (5.22) and (5.24) are vestiges of a lost paradise: the classical domain $\mathcal{D}^{(3)}$ and its Kählerian attributes. Property (5.22) also exists for the anti-de Sitterian–Cartan sections (3.9) and (3.12),

$$p : \mathrm{Sp}(4,\mathbf{R})/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)) \longrightarrow \mathrm{Sp}(4,\mathbf{R}) \tag{5.25}$$

$$\Pi : \mathrm{SO}_0(3,2)/\mathrm{SO}(3) \times \mathrm{SO}(2) \longrightarrow \mathrm{SO}_0(3,2) \tag{5.26}$$

namely

$$(p(z))^{-1} = p(-z) \tag{5.27}$$

$$(\Pi(X))^{-1} = \Pi(-X). \tag{5.28}$$

To understand the origin of (5.24), we use the $(\boldsymbol{q},\boldsymbol{p})$ first-order parametrization of the coset (5.26) afforded by equations (5.15) and (5.16), or alternatively from equation (3.14),

$$\mathcal{D}^{(3)} \ni \boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y} = \frac{mc\kappa\boldsymbol{q} + \mathrm{i}\boldsymbol{p}}{p_0 + mc} + \mathrm{o}(\kappa^2). \tag{5.29}$$

The fact that the imaginary part $\boldsymbol{y}$ has no first-order expansion term will be given an *a posteriori* justification in section 7. Note that it could be inferred from the global $(\boldsymbol{q},\boldsymbol{p})$-coordinatization of $\mathcal{D}^{(3)}$,

$$\begin{aligned} \boldsymbol{z} &= \frac{\left[mc\,\sinh(\kappa q) + \mathrm{i}(\hat{q}\cdot\boldsymbol{p})(\cosh(\kappa q)-1)\right]\hat{q} + \mathrm{i}\boldsymbol{p}}{\left[(p_0\cosh(\kappa q)+mc)^2 - (\hat{q}\cdot\boldsymbol{p})^2\sinh^2(\kappa q)\right]^{1/2}} \\ \hat{q} &= \frac{\boldsymbol{q}}{q} \qquad \text{and} \qquad q = \|\boldsymbol{q}\| \end{aligned} \tag{5.30}$$

in agreement with the $\mathrm{Sp}(4,\mathbf{R})$-Cartan factorization and its Poincaré limit. The infinitesimal transformation corresponding to equation (5.29) reads

$$\begin{pmatrix} \mathrm{d}\boldsymbol{x} \\ \mathrm{d}\boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \dfrac{mc\kappa}{p_0+mc}1_3 & -\dfrac{mc\kappa\boldsymbol{q}\boldsymbol{p}^{\mathrm{t}}}{p_0(p_0+mc)^2} \\ 0 & \dfrac{1}{p_0+mc}1_3 - \dfrac{\boldsymbol{p}\boldsymbol{p}^{\mathrm{t}}}{p_0(p_0+mc)^2} \end{pmatrix} \begin{pmatrix} \mathrm{d}\boldsymbol{q} \\ \mathrm{d}\boldsymbol{p} \end{pmatrix} + \mathrm{o}(\kappa^2). \tag{5.31}$$

The leading term of the Jacobian is $m^4c^4\kappa^3/p_0(p_0+mc)^6$. Other approximation formulae are useful:

$$1 + \boldsymbol{z}\bar{\boldsymbol{z}} = \frac{2mc}{p_0+mc} - \mathrm{i}\frac{2mc\kappa}{(p_0+mc)^2}\boldsymbol{q}\times\boldsymbol{p} + \mathrm{o}(\kappa^2) \tag{5.32}$$

$$\det(1+\boldsymbol{z}\bar{\boldsymbol{z}}) = \frac{4m^2c^2}{(p_0+mc)^2} + \mathrm{o}(\kappa^2). \tag{5.33}$$

It follows the leading term for the invariant volume element (3.21):

$$\mathrm{d}\mu(\boldsymbol{z}) \simeq \frac{3\kappa^3}{8\pi^3m^2c^2}\mathrm{d}^3\boldsymbol{q}\frac{\mathrm{d}^3\boldsymbol{p}}{p_0} \propto \mathrm{d}\mu_s(\boldsymbol{q},\boldsymbol{p}). \tag{5.34}$$

The approximate Kählerian metric and 2-form are also interesting,

$$\mathrm{d}s^2 = -\frac{3}{2m^2c^2}\left[(\mathrm{d}p_0)^2 - \|\mathrm{d}\boldsymbol{p}\|^2\right] + \mathrm{o}(\kappa^2) \tag{5.35}$$

$$\omega = -\frac{3\kappa}{2mc}\left[\sum_i \mathrm{d}q^i\wedge\mathrm{d}p^i - \frac{\boldsymbol{p}\cdot\mathrm{d}\boldsymbol{q}\wedge\boldsymbol{p}\cdot\mathrm{d}\boldsymbol{p}}{p_0(p_0+mc)} + 3\frac{\boldsymbol{q}\cdot\mathrm{d}\boldsymbol{p}\wedge\boldsymbol{p}\cdot\mathrm{d}\boldsymbol{p}}{p_0(p_0+mc)}\right]. \tag{5.36}$$

Let us say more about the invariance of the above three expressions under the Poincaré action. An arbitrary element $(q,\Lambda)\in\mathcal{P}_+^\uparrow(3,1)$ may be factorized either as

$$(q,\Lambda) = (q_s', L_{\boldsymbol{p}})\left(\left(q_0 - \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{p_0+mc},\boldsymbol{0}\right), R_\Lambda\right) \tag{5.37}$$

according to the left coset Γ_{l} or as

$$(q,\Lambda) = \left(\left(q_0 - \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{p_0+mc},\boldsymbol{0}\right), R_\Lambda\right)(q_s'', L_{\boldsymbol{p}''}) \tag{5.38}$$

according to Γ_r. In the former case, $\Lambda = L_p R_\Lambda$, $R_\Lambda \in SO(3)$, and $q'_s \boxminus (q' \cdot p/(p_0 + mc), q')$ with

$$q' = \overrightarrow{L_{-p} q} \\ = q + \frac{q \cdot p}{mc(p_0 + mc)} p - q_0 \frac{p}{mc}. \tag{5.39}$$

In the latter case, q''_s is defined by

$$q''_s = \left(\frac{q \cdot p}{p_0 + mc}, R^t_\Lambda q \right) \tag{5.40}$$

and $p'' = R^t_\Lambda p$. Hence, the section (5.18) is valid for both Γ_l and Γ_r and elements of both cosets can be parametrized by $(q, p) \in \mathbf{R}^6$ according to β_s. The left and right actions of $\mathcal{P}^\uparrow_+(3,1)$ on Γ_l and Γ_r, respectively, are similar. On Γ_l the action is

$$(q, p) \longmapsto (q', p') = (a, \Lambda)(q, p) \tag{5.41}$$

where

$$q' = R_\Lambda R_{\overrightarrow{R^t_\Lambda k}, p} q + \overrightarrow{L_{-\overrightarrow{\Lambda p}} a} \tag{5.42}$$

$$p' = \overrightarrow{\Lambda p}. \tag{5.43}$$

Here, the rotation $R_{k,k'}$ appears in the Cartan decomposition of the product of two boosts.

$$L_k L_{k'} = L_{\overrightarrow{L_k k'}} R_{k,k'}. \tag{5.44}$$

On the other hand, the action of $\mathcal{P}^\uparrow_+(3,1)$ on Γ_r is given by

$$q'' = R^t_\Lambda R_{\overrightarrow{L_{-k} S}, k} (q + \overrightarrow{L_p a}) \tag{5.46}$$

$$p'' = \overrightarrow{S \Lambda^{-1} \mathrm{S} p} \tag{5.47}$$

where S is the spatial inversion. The invariance of (5.24), (5.35) and (5.36) holds by reference to the above left and right actions.

6. Anti-de Sitter–Poincaré contraction: operatorial aspects

We now turn to the task of contracting the representation $T^{(E_0,s)}$, given in (4.6), at the level of its infinitesimal generators (4.7) and (4.8). At this end, we begin by rescaling *à la* Inönü–Wigner [2], the generators which are not Lorentz:

$$\widetilde{L}_{50} \equiv \kappa L_{50} \qquad \widetilde{L}_{5i} \equiv \kappa L_{5i} \qquad \widetilde{L}_{ij} \equiv L_{ij} \qquad \text{and} \qquad \widetilde{L}_{0i} \equiv L_{0i}. \tag{6.1}$$

Formally, the new commutation rules become the Poincaré ones when $\kappa = 0$. However things are not so simple, since a radical change can occur for the infinitesimal operators originally acting on the Bargmann–Hilbert space $\mathcal{F}^{(E_0,s)}$. One should study carefully what happens, for the spaces of analytic functions and the operator domains, at the topological and functional analysis levels. Such a mathematical development is presented elsewhere [18]. Here we describe rather heuristically the contraction procedure in the spirit of [19], and we deal with the scalar case only. The non-zero spin case is considered in the appendix.

Taking into account the parametrization (5.30), the asymptotic behaviour ($\kappa \sim 0$) of the differential operators ∇_x and ∇_y is the following:

$$\nabla_x \sim \frac{1}{\kappa}\frac{p_0+m}{m}\nabla_q - \kappa\frac{p_0}{m}p\cdot\nabla_p \tag{6.2a}$$

$$\nabla_y \sim (p_0+m)\nabla_p + \frac{p}{m}\left(q\cdot\nabla_q + p\cdot\nabla_p\right). \tag{6.2b}$$

Before writing out the explicit form of the infinitesimal generators in the limit $\kappa \to 0$, we would like to add some comments. The representation parameter E_0 is a pure number. The contraction procedure consists in taking the limits $E_0 \to \infty$ and $\kappa \to 0$, while keeping the product κE_0 finite and proportional to m, the rest mass of the limiting Poincaré elementary system. However, the three (fundamental) constants κ, m and c appearing in the formalism are not enough to build up pure numbers. We should be aware that the Fock–Bargmann–Hilbert spaces are dealt within a quantum context, characterized by action-dimensioned physical quantities of order $\hbar$. Now, the unique dimensionless combination of these four constants is the parameter

$$\xi = \frac{\hbar\kappa}{mc} \tag{6.3}$$

and the contraction condition $\kappa E_0 \propto m$ can be replaced by

$$E_0 = \xi^{-1}. \tag{6.4}$$

Actually we might consider the quantum anti-de Sitter parameter E_0 as a meromorphic function of ξ with simple pole $\xi = 0$,

$$E_0 = E_0(\xi) = \xi^{-1} + \sum_{n\geqslant 0} e_n\xi^n \tag{6.5}$$

where the $e_0, e_1, \ldots, e_n, \ldots$ are pure numbers. So considering (6.2) and (6.5), we are led to the following asymptotic behaviour for the infinitesimal generators of (4.8):

$$\widetilde{M}_{50} = \mathrm{i}\frac{p}{2mc}\cdot\nabla_q + \frac{mc}{\hbar} + \mathrm{o}(\xi) \tag{6.6a}$$

$$\widetilde{M}_{5i} = \frac{\mathrm{i}}{2}\frac{\partial}{\partial q^i} - \frac{p^i}{(p_0+mc)}\left(\frac{mc}{\hbar} + \frac{\mathrm{i}}{2mc}p\cdot\nabla_q\right) + \mathrm{o}(\xi) \tag{6.6b}$$

$$\widetilde{M}_{ij} = \xi^{-1}\frac{\hbar}{2m^2c^2}\left(p_i\frac{\partial}{\partial q^j} - p_j\frac{\partial}{\partial q^i}\right) -$$

$$\frac{\mathrm{i}}{2}\left(q_i\frac{\partial}{\partial q^j}-q_j\frac{\partial}{\partial q^i}+p_i\frac{\partial}{\partial p^j}-p_j\frac{\partial}{\partial p^i}\right)+\mathrm{o}(\xi) \tag{6.6c}$$

$$\begin{aligned}\widetilde{M}_{0i}=-\mathrm{i}\xi^{-1}\frac{\hbar}{mc}&\left(-\mathrm{i}\frac{p_0}{2mc}\frac{\partial}{\partial q^i}-\mathrm{i}\frac{p^i}{2mc(p_0+mc)}\boldsymbol{p}\cdot\boldsymbol{\nabla}_q-\frac{mcp^i}{\hbar(p_0+mc)}\right)\\&+\frac{\mathrm{i}}{2}p_0\frac{\partial}{\partial p^i}+\frac{\mathrm{i}}{2}\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{(p_0+mc)}\frac{\partial}{\partial q^i}+\frac{m^2c^2q^i}{\hbar(p_0+mc)}\left(1+\frac{\mathrm{i}\hbar}{2m^2c^2}\boldsymbol{p}\cdot\boldsymbol{\nabla}_q\right)\\&-\mathrm{i}e_0\frac{p^i}{p_0+mc}+\mathrm{o}(\xi).\end{aligned} \tag{6.6d}$$

In the limit $\xi=0$, singular terms appear in the expression of the generators of the subgroup SO(3.1), namely $\widetilde{M}_{ij}$ and $\widetilde{M}_{0i}$. We shall need a specific notation for them, whereas we adopt Poincaré-like notations for the remnant terms;

$$\widetilde{M}_{50}=\widetilde{P}_0+\mathrm{o}(\xi) \tag{6.7a}$$
$$\widetilde{M}_{5i}=\widetilde{P}_i+\mathrm{o}(\xi) \tag{6.7b}$$
$$\widetilde{M}_{ij}=\xi^{-1}\frac{\mathrm{i}}{mc}\epsilon_{ij}{}^{k}\widetilde{\Sigma}_k+\epsilon_{ij}{}^{k}\widetilde{J}_k+\mathrm{o}(\xi) \tag{6.7c}$$
$$\widetilde{M}_{0i}=\xi^{-1}\frac{\mathrm{i}}{mc}\widetilde{\Pi}_i+\widetilde{K}_i+\mathrm{o}(\xi). \tag{6.7d}$$

One should first note that the operator $\widetilde{K}_i$ will be Hermitian only if $e_0=0$. On the other hand, the operators $\widetilde{P}_0,\widetilde{P}_i,\widetilde{J}_i$ and $\widetilde{K}_i$ $(i=1,2,3)$, exactly obey the usual Poincaré commutation rules if we make the rescaling [19],

$$\frac{1}{2}\frac{\partial}{\partial q^i}\longmapsto\frac{\partial}{\partial q^i}\qquad\text{and}\qquad\frac{1}{2}\frac{\partial}{\partial p^i}\longmapsto\frac{\partial}{\partial p^i}. \tag{6.8}$$

Such a seemingly artificial rescaling can be shown to be unnecessary if the contraction procedure is performed in more rigorous mathematical settings [18]; we shall come back to this point in section 7. Finally, by multiplying each one of the latter generators by $\hbar$, one obtains the Poincaré generators under their definitive form,

$$P_0=\mathrm{i}\frac{\hbar}{mc}\boldsymbol{p}\cdot\boldsymbol{\nabla}_q+mc \tag{6.9a}$$
$$P_i=\mathrm{i}\hbar\frac{\partial}{\partial q^i}-\mathrm{i}\frac{\hbar}{mc}\frac{p^i}{p_0+mc}\boldsymbol{p}\cdot\boldsymbol{\nabla}_q-\frac{mcp^i}{p_0+mc} \tag{6.9b}$$
$$J_i=-\frac{\mathrm{i}}{2}\hbar\epsilon_i{}^{jk}\left(q_j\frac{\partial}{\partial q^k}-q_k\frac{\partial}{\partial q^j}+p_j\frac{\partial}{\partial p^k}-p_k\frac{\partial}{\partial p^j}\right) \tag{6.9c}$$
$$K_i=\mathrm{i}\hbar p_0\frac{\partial}{\partial p^i}+\mathrm{i}\hbar\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{p_0+mc}\frac{\partial}{\partial q^i}+\frac{m^2c^2q^i}{p_0+mc}\left(1+\frac{\mathrm{i}\hbar}{m^2c^2}\boldsymbol{p}\cdot\boldsymbol{\nabla}_q\right). \tag{6.9d}$$

The remaining terms in (6.7c) and (6.7d) become infinite at the limit $\xi\to 0$. We remove these singularities by imposing on the space of functions on the $(\boldsymbol{q},\boldsymbol{p})$ phase space, the following constraints:

$$\forall\Phi(\boldsymbol{q},\boldsymbol{p})\qquad\begin{cases}(\boldsymbol{\Sigma}\Phi)(\boldsymbol{q},\boldsymbol{p})=0 & (6.10a)\\(\boldsymbol{\Pi}\Phi)(\boldsymbol{q},\boldsymbol{p})=0 & (6.10b)\end{cases}$$

where

$$\Sigma_i = -\mathrm{i}\frac{\hbar}{mc}\epsilon_{ijk}p^j\frac{\partial}{\partial q_k} \tag{6.11a}$$

$$\Pi_i = \mathrm{i}\frac{\hbar}{mc}p_0\frac{\partial}{\partial q^i} - \mathrm{i}\frac{\hbar}{mc}\frac{p^i}{p_0 + mc}\boldsymbol{p}\cdot\boldsymbol{\nabla}_q - mc\frac{p^i}{p_0 + mc}. \tag{6.11b}$$

Equations (6.10) are typical polarization conditions, usually encountred in geometric quantization [18, 27]. For this reason, we shall call $\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$ polarization operators.

Besides the familiar Poincaré commutation rules, we have those involving the polarization operators,

$$[\Sigma_i, \Sigma_j] = [\Pi_i, \Pi_j] = [\Sigma_i, \Pi_j] = 0 \tag{6.12a}$$

$$[\boldsymbol{\Sigma}, P_\mu] = [\boldsymbol{\Pi}, P_\mu] = 0 \tag{6.12b}$$

$$[J_i, \Sigma_j] = \mathrm{i}\hbar\epsilon_{ij}{}^k\Sigma_k \qquad [J_i, \Pi_j] = \mathrm{i}\hbar\epsilon_{ij}{}^k\Pi_k \tag{6.12c}$$

$$[K_i, \Sigma_j] = \mathrm{i}\hbar\epsilon_{ij}{}^k\Pi_k \qquad [K_i, \Pi_j] = \mathrm{i}\hbar\epsilon_{ij}{}^k\Sigma_k. \tag{6.12d}$$

Therefore the generators $\boldsymbol{J}, \boldsymbol{K}, P_0, \boldsymbol{P}, \boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$ form a 16-dimensional Lie algebra that is the semi-direct sum of so(3,1) and a 10-dimensional Abelian algebra,

$$\{\boldsymbol{J}, \boldsymbol{K}\} \ominus \{P_0, \boldsymbol{P}, \boldsymbol{\Sigma}, \boldsymbol{\Pi}\}. \tag{6.13}$$

This extended Poincaré algebra has a second-order invariant (Casimir) operator, which is identically equal to m^2c^2,

$$I_2 = P_\mu P^\mu - \Pi_i\Pi^i + \Sigma_i\Sigma^i \equiv m^2c^2\mathrm{Id}. \tag{6.14}$$

The functional space at $\xi = 0$ limit results from the polarization conditions (6.10); therefore, on this space, I_2 becomes the Poincaré Casimir operator $P_\mu P^\mu$ and (6.14) becomes the Klein–Gordon equation,

$$(P_\mu P^\mu - m^2c^2)\Phi(q,p) = 0 \tag{6.15}$$

for $\Phi(\boldsymbol{q}, \boldsymbol{p})$ given by (6.10). Actually the two conditions (6.10a) and (6.10b) reduce to one by noting that,

$$(\boldsymbol{\Pi}\Phi)(q,p) = 0 \Longrightarrow (\boldsymbol{\Sigma}\Phi)(q,p) = 0. \tag{6.16}$$

This is directly deduced from the explicit formulae of Σ_i and Π_i given in (6.11).

One can see, by solving the polarization condition (6.10b), that the carrier space of functions Φ of the representation is no longer a subspace of $L^2(\mathbf{R}^6, \mathrm{d}^3q\mathrm{d}^3p/p_0)$; in other words we lose the square integrability on the entire phase space, when ξ reaches its limiting value 0. In fact, when solving (6.10b) one obtains

$$\Phi(\boldsymbol{q}, \boldsymbol{p}) = \phi(\boldsymbol{p})\exp(\mathrm{i}\hbar q_s^\mu p_\mu) \qquad q_s = \left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{p_0 + m}, \boldsymbol{q}\right) \tag{6.17}$$

where q_s is the *Cartan–anti-de Sitterian section* (5.17). The energy and momentum operators, P_0 and $\boldsymbol{P}$, are diagonal on the space of such functions (6.17). On the other hand, the exponential factor in (6.17) is transparent with respect to the Lorentz action. Thus the action of the Poincaré operators (6.9) passes through this modulus-one factor to become the well known infinitesimal Wigner–Poincaré action on functions $\phi(\boldsymbol{p})$ defined on the mass-shell [5];

$$(P_0\phi)(\boldsymbol{p}) = p_0\phi(\boldsymbol{p}) \tag{6.18a}$$

$$\left(\vec{P}\phi\right)(\boldsymbol{p}) = \boldsymbol{p}\phi(\boldsymbol{p}) \tag{6.18b}$$

$$(J_i\phi)(\boldsymbol{p}) = -\mathrm{i}\frac{\hbar}{2}\epsilon_i{}^{jk}\left(p_j\frac{\partial}{\partial p^k} - p_k\frac{\partial}{\partial p^j}\right)\phi(\boldsymbol{p}) \tag{6.18c}$$

$$(K_i\phi)(\boldsymbol{p}) = \mathrm{i}\hbar p_0\frac{\partial}{\partial p^i}\phi(\boldsymbol{p}). \tag{6.18d}$$

Thus the Wigner representation on $L^2\left(\mathcal{V}_m^+, \mathrm{d}^3\boldsymbol{p}/p_0\right)$ is recovered by just imposing the functions $\phi(\boldsymbol{p})$ of (6.17) be square integrable with respect to $\mathrm{d}^3\boldsymbol{p}/p_0$.

7. Discussion

In section 5 the first-order expansion of $\boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y} \in \mathcal{D}^{(3)}$ exhibits no first-order term in the asymptotic form of $\boldsymbol{y}$ (see (5.29)). That was justified from the global coordinatization of $\mathcal{D}^{(3)}$ given in (5.30). Asymptotically, the latter provides exactly (5.29). Had we chosen another global parametrization of $\mathcal{D}^{(3)}$, which in its asymptotic form can be written in general as

$$\boldsymbol{x} = \frac{m\kappa\boldsymbol{q}}{p_0 + m} + \mathrm{o}(\kappa^2) \tag{7.1a}$$

$$\boldsymbol{y} = \frac{\boldsymbol{p}}{p_0 + m} + \kappa\boldsymbol{f}(\boldsymbol{q}, \boldsymbol{p}) + \mathrm{o}(\kappa^2) \tag{7.1b}$$

where $\boldsymbol{f}$ is an arbitrary vector function analytic in $\boldsymbol{q}$ and $\boldsymbol{p}$, the infinitesimal generators (6.6) would aquire extra terms involving $\boldsymbol{f}$ and its derivatives. It is then easy to show that their non-singular (and rescaled) parts as in (6.7), (6.8) and (6.9), obey the usual Poincaré commutation rules only if $\boldsymbol{f} = 0$. So the parametrization must be such that no term linear in κ appears in the asymptotic expansion of $\boldsymbol{y}$.

The procedure of contraction of the discrete series performed in this work presents some problems; these are essentially two: the non-natural rescaling used in (6.8) and the singular terms appearing in the contraction of the infinitesimal generators. The interpretation of the latter as polarization operators draws one naturally, in order to explain the origin of the above-mentioned problems, to put the present work into the framework of geometric quantization. This has been done in [18]. There, no such problems appear and an explanation to the present ones is given. Actually it is shown that they have the same origin. Singular terms appear naturally when one contracts the holomorphic part of the prequantized representation, since it carries implicitly a polarization condition, i.e. the holomorphic condition. Then those singular terms appear as the Poincaré counterparts of the anti-de Sitterian holomorphic condition. In [18] the contraction is performed at the prequantized level, i.e. taking into account the holomorphic and the anti-holomorphic parts, that solves the problem of singular terms and that of the rescaling since no holomorphic condition is considered and each part contributes by one half in the contracted generators. Then the construction is completed by adding to the resulting Poincaré generators the $\kappa \to 0$ limit of the anti-de Sitter polarization (the holomorphic condition), which is interpreted as the Poincaré polarization. One thus obtains the Poincaré quantum elementary system.

Appendix

In this appendix we give the complete form of equations (4.7) and (4.8), (6.6), (6.9) and (6.18) by displaying their spin parts. We start by equations (4.7) and (4.8). For the calculations we make use essentially of equation (4.6).

First let us introduce the $(2s+1) \times (2s+1)$ matrices $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ given respectively by

$$(\mathcal{S}_1)_{m\,m'} = \tfrac{1}{2}\sqrt{(s+m)(s-m+1)}\,\delta_{m,m'+1} + \tfrac{1}{2}\sqrt{(s-m)(s+m+1)}\,\delta_{m,m'-1} \tag{A.1}$$

$$(\mathcal{S}_2)_{m\,m'} = \frac{1}{2\mathrm{i}}\sqrt{(s+m)(s-m+1)}\,\delta_{m,m'+1} - \frac{1}{2\mathrm{i}}\sqrt{(s-m)(s+m+1)}\,\delta_{m,m'-1} \tag{A.2}$$

$$(\mathcal{S}_3)_{m\,m'} = m\delta_{m,m'} \tag{A.3}$$

for m and m' integers such $-s \leqslant m, m' \leqslant s$; they realize the spin s representation of the Lie algebra of $\mathrm{SU}(2)$, i.e.

$$[\mathcal{S}_i, \mathcal{S}_j] = -\mathrm{i}\,\epsilon_{ij}{}^{k}\mathcal{S}_k \qquad i,j,k \in \{1,2,3\}. \tag{A.4}$$

In terms of these matrices the complete generators of (4.7) are now displayed,

$$L_{50} = (\boldsymbol{z}\cdot\boldsymbol{\nabla}_z + E_0)\,\mathrm{Id} \qquad \boldsymbol{\nabla}_z = \left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^3}\right) \tag{A.5}$$

$$L_{12} = -\mathrm{i}\left(z_1\frac{\partial}{\partial z^2} - z_2\frac{\partial}{\partial z^1}\right)\mathrm{Id} - \mathcal{S}_3 \tag{A.6a}$$

$$L_{23} = -\mathrm{i}\left(z_2\frac{\partial}{\partial z^3} - z_3\frac{\partial}{\partial z^2}\right)\mathrm{Id} - \mathcal{S}_1 \tag{A.6b}$$

$$L_{31} = -\mathrm{i}\left(z_3\frac{\partial}{\partial z^1} - z_1\frac{\partial}{\partial z^3}\right)\mathrm{Id} - \mathcal{S}_2 \tag{A.6c}$$

$$L_{51} = \mathrm{i}\left(\frac{1+\boldsymbol{z}\cdot\boldsymbol{z}}{2}\frac{\partial}{\partial z^1} - z^1(E_0 + s + \boldsymbol{z}\cdot\boldsymbol{\nabla}_z)\right)\mathrm{Id} + z^2\mathcal{S}_3 - z^3\mathcal{S}_2 \tag{A.7a}$$

$$L_{52} = \mathrm{i}\left(\frac{1+\boldsymbol{z}\cdot\boldsymbol{z}}{2}\frac{\partial}{\partial z^2} - z^2(E_0 + s + \boldsymbol{z}\cdot\boldsymbol{\nabla}_z)\right)\mathrm{Id} - z^1\mathcal{S}_3 + z^3\mathcal{S}_1 \tag{A.7b}$$

$$L_{53} = \mathrm{i}\left(\frac{1+\boldsymbol{z}\cdot\boldsymbol{z}}{2}\frac{\partial}{\partial z^3} - z^3(E_0 + s + \boldsymbol{z}\cdot\boldsymbol{\nabla}_z)\right)\mathrm{Id} + z^1\mathcal{S}_2 - z^2\mathcal{S}_1 \tag{A.7c}$$

$$L_{01} = -\left(\frac{1-z\cdot z}{2}\frac{\partial}{\partial z^1} + z^1(E_0 + s + z\cdot\nabla_z)\right)\mathrm{Id} - \mathrm{i}z^2\mathcal{S}_3 + \mathrm{i}z^3\mathcal{S}_2 \tag{A.8a}$$

$$L_{02} = -\left(\frac{1-z\cdot z}{2}\frac{\partial}{\partial z^2} + z^2(E_0 + s + z\cdot\nabla_z)\right)\mathrm{Id} + \mathrm{i}z^1\mathcal{S}_3 - \mathrm{i}z^3\mathcal{S}_1 \tag{A.8b}$$

$$L_{03} = -\left(\frac{1-z\cdot z}{2}\frac{\partial}{\partial z^3} + z^3(E_0 + s + z\cdot\nabla_z)\right)\mathrm{Id} - \mathrm{i}z^1\mathcal{S}_2 + \mathrm{i}z^2\mathcal{S}_1 \tag{A.8c}$$

where Id is the $(2s+1)\times(2s+1)$ identity matrix.

Since the spin contributions are linear in the components of z, no new singular terms will appear when carrying out the contraction. Thus the polarization operators are unchanged, they only become matrix-valued operators. This is a mere consequence of equation (5.29). Moreover, both L_{50} and the L_{ij}s are free from new κ-dependent terms, so their contraction is straightforward. More precisely, L_{50} has no spin part and thus its contraction will provide the same result as in section 6, namely equation (6.9a). For the L_{ij}s the spin parts are just the (κ-independent) spin-s matrix representatives of the Lie algebra of $\mathrm{SU}(2)$. The contraction will not affect these parts. The contracted rotation generators are then

$$J_i^{(s)} = J_i + \hbar\mathcal{S}_i \qquad i\in\{1,2,3\} \tag{A.10}$$

the J_is and the $\mathcal{S}_i$s being given respectively in (6.9c) and (A.1–A.4) and they are the components of the 3-vectors J and $\vec{\mathcal{S}}$ respectively, with $J_i = -J^i$ and $\mathcal{S}_i = -\mathcal{S}^i$.

The contractions of the L_{5i}s and L_{0i}s need simple calculations. We start by recalling from (5.29) the leading term in the expansion of z,

$$z = \mathrm{i}\frac{p}{p_0 + mc} + \mathrm{o}(\kappa). \tag{A.11}$$

It is then easy to see that the leading term of $\widetilde{L}_{5i} \equiv \kappa L_{5i}$ is exactly that of $\widetilde{M}_{5i}$ given in (6.6b). The contracted generators are then exactly those found in (6.9b). Finally for the L_{0i}s a direct calculation based on (A.11) and (A.8) gives the spin-s counterpart of (6.6d),

$$\widetilde{L}_{01} = \widetilde{M}_{01} + \frac{1}{p_0+mc}(p^2\mathcal{S}_3 - p^3\mathcal{S}_2) + \mathrm{i}s\frac{p^1}{p_0+mc} + \mathrm{o}(\xi) \tag{A.12a}$$

$$\widetilde{L}_{02} = \widetilde{M}_{02} + \frac{1}{p_0+mc}(p^3\mathcal{S}_1 - p^1\mathcal{S}_3) + \mathrm{i}s\frac{p^2}{p_0+mc} + \mathrm{o}(\xi) \tag{A.12b}$$

$$\widetilde{L}_{03} = \widetilde{M}_{03} + \frac{1}{p_0+mc}(p^1\mathcal{S}_2 - p^2\mathcal{S}_1) + \mathrm{i}s\frac{p^3}{p_0+mc} + \mathrm{o}(\xi) \tag{A.12c}$$

the $\widetilde{M}_{0i}$s being given in (6.6d). After contraction they become the Poincaré generators of boosts with spin s. In a more compact form, we have

$$K^{(s)} = K + \mathrm{i}\hbar s\frac{p}{p_0+mc} + \hbar\frac{(p\times\vec{\mathcal{S}})}{p_0+mc} \tag{A.13}$$

where the components of K are given in (6.9d).

Since the polarization operators are unchanged when considering spin, the unique modification in (6.17) consists in replacing the scalar functions by $(2s+1)$-vector valued functions, (see for instance equation (4.1)). Moreover, the spin counterparts of equations (6.18a–d) are easy to write down according to the latter observation and to the equations (A.10) and (A.13). We then obtain the Wigner representation of the Poincaré group with arbitrary spin.

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